More l'Hopital's Rule

Example 1: $(\infty - \infty)$

Evaluate $\displaystyle\lim_{x\to 2}\left(\frac{1}{(x-2)^2} - \cot^2(x-2)\right)$

$$\lim_{x\to 2}\frac{1}{(x-2)^2} = \infty$$

$$\lim_{x\to 2}\cot^2(x-2) = \lim_{x\to 2}\frac{\cos^2(x-2)}{\sin^2(x-2)}$$

$$= \infty$$

Rewrite the entire limit as a quotient.

$$\frac{1}{(x-2)^2} - \cot^2(x-2)$$

$$= \frac{1 - ((x-2)\cot(x-2))^2}{(x-2)^2}$$

Now $\lim\limits_{x\to 2}((x-2)\cot(x-2))$

$$= \lim_{x\to 2}\left(\cos(x-2)\,\frac{(x-2)}{\sin(x-2)}\right)$$

$$= \lim_{x\to 2}\cos(x-2)\lim_{x\to 2}\frac{(x-2)}{\sin(x-2)}$$

$= 1 \cdot 1 = 1$, so we have $\frac{0}{0}$.

Use l'Hopital's Rule.

$$\lim_{x\to 2} \frac{1-((x-2)\cot(x-2))^2}{(x-2)^2}$$

$$\overset{l'H}{=} \lim_{x\to 2} \frac{-\cancel{2}\cancel{(x-2)}\cot(x-2)\left(\cot(x-2)-(x-2)\cot(x-2)\csc(x-2)\right)}{\cancel{2}\cancel{(x-2)}}$$

$$= \lim_{x\to 2} -\cot^2(x-2)\left(1-(x-2)\csc(x-2)\right)$$

Since $\lim\limits_{x\to 2} (x-2)\csc(x-2) = \lim\limits_{x\to 2} \frac{x-2}{\sin(x-2)} = 1$,

this limit is $\infty \cdot 0$, so rewrite as a quotient.

$$\lim_{x \to 2} -\cot^2(x-2)\left(1-(x-2)\csc(x-2)\right)$$

$$= \lim_{x \to 2} -\frac{\cos^2(x-2)}{\sin^2(x-2)}\left(1-\frac{x-2}{\sin(x-2)}\right)$$

$$= \lim_{x \to 2} -\frac{\cos^2(x-2)}{\sin^2(x-2)}\left(\frac{\sin(x-2)-(x-2)}{\sin(x-2)}\right)$$

$$= \lim_{x \to 2} \frac{-\cos^2(x-2)\left(\sin(x-2)-(x-2)\right)}{\sin^3(x-2)}$$

Now use l'Hopital's rule.

$$\lim_{x \to 2} \frac{-\cos^2(x-2)\left(\sin(x-2)-(x-2)\right)}{\sin^3(x-2)}$$

$$\overset{\text{l'H}}{=} \lim_{x \to 2} \frac{\left(2\cos(x-2)\sin(x-2)\left(\sin(x-2)-(x-2)\right) - \cos^2(x-2)\left(\cos(x-2)-1\right)\right)}{3\sin^2(x-2)\cos(x-2)}$$

$$= \frac{1}{0} \quad \boxed{\text{does not exist}}$$

Example 2: $(1^{\pm\infty})$

Compute $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$

This is 1^{∞} in the limit, which is NOT one. Do more work!

$$\left(1+\frac{1}{x}\right)^{x} = e^{\ln\left(\left(1+\frac{1}{x}\right)^{x}\right)}$$

$$= e^{x\ln\left(1+\frac{1}{x}\right)} \quad \text{(log rules)}$$

Figure out the limit of the exponent, then raise e to that power.

$\lim_{x\to\infty} x\cdot \ln\left(1+\frac{1}{x}\right)$ is

an indeterminate product $(\infty \cdot 0)$;

rewrite as a quotient, use

l'Hopital's rule.

$$x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$$

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\ln(1+\frac{1}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\ln(1+x^{-1})}{x^{-1}}$$

$$\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+x^{-1}}\cancel{(-x^{-2})}}{\cancel{-x^{-2}}}$$

$$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1.$$

So the final answer is $e^1 = \boxed{e}$

Example 3: (0^0)

Determin $\lim_{x \to 1^-} (\ln(2-x))^{(x-1)}$

Another exponent, use $e^{\ln}$ trick again:

$$(\ln(2-x))^{x-1} = e^{\ln\left((\ln(2-x))^{(x-1)}\right)}$$

$$= e^{(x-1)\ln(\ln(2-x))}$$

Figure out $\lim_{x \to 1^-} (x-1)\ln(\ln(2-x))$,

then take e to that power.

Rewrite as a quotient.

$$(x-1)\ln(\ln(2-x)) = \frac{\ln(\ln(2-x))}{\left(\frac{1}{x-1}\right)}$$

$$\lim_{x\to 1^-} \frac{\ln(\ln(2-x))}{\left(\frac{1}{x-1}\right)} = \lim_{x\to 1^-} \frac{\ln(\ln(2-x))}{(x-1)^{-1}}$$

$$\overset{l'H}{=} \lim_{x\to 1^-} \frac{\frac{1}{\ln(2-x)} \frac{d}{dx}(\ln(2-x))}{-(x-1)^{-2}}$$

$$= \lim_{x\to 1^-} \frac{\frac{1}{\ln(2-x)} \cdot \frac{1}{x-2}}{-(x-1)^{-2}}$$ since 5

$$\frac{d}{dx}\left(\ln(2-x)\right) = \frac{1}{2-x}\,\frac{d}{dx}(2-x)$$

$$= \frac{1}{2-x}(-1)$$

$$= \frac{1}{2-x}\,\frac{1}{(-1)}$$

$$= \frac{1}{x-2}$$

Back to the limit...

$$\lim_{x \to 1^-} \frac{\frac{1}{\ln(2-x)} \cdot \frac{1}{x-2}}{-(x-1)^{-2}}$$

$$= \lim_{x \to 1^-} \frac{-(x-1)^2}{\ln(2-x)(x-2)}, \text{ still } \frac{0}{0}, \text{ so}$$

$$\overset{\text{L'H}}{=} \lim_{x \to 1^-} \frac{-2(x-1)}{\ln(2-x) + \cancel{(x-2)} \cdot \left(\frac{1}{\cancel{x-2}}\right)}$$

$$= \lim_{x \to 1^-} \frac{-2(x-1)}{\ln(2-x) + 1}$$

$$= \frac{0}{1} = 0.$$

Final answer: $e^0 = \boxed{1}$

Example 4: (∞^0)

Find $\lim_{x\to\infty} x^{(2/x)}$.

Again, use $e^{\ln}$ trick.

$$e^{\ln\left(x^{(2/x)}\right)} = e^{\frac{2\ln(x)}{x}}.$$

Take the limit of the exponent.

$$\lim_{x\to\infty} \frac{2\ln(x)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2\cdot\frac{1}{x}}{1} = 0$$

Final answer: $e^0 = \boxed{1}$

Applications of Exponentials

(Section 6.5)

Exponential Growth and Decay

You have \$10,000 to invest in a campaign fund for Newt Gingrich. There are four choices:

Compounding Periods

1) Yearly
2) Quarterly
3) Monthly
4) Mystery option.

Newt will give you the great rate of 10% interest. Which of these choices gives the most money after a year?

1) Yearly:

$\$10,000 + (.10) \cdot (\$10,000)$

$= \$11,000$ after one year.

2) Quarterly: divide the interest rate by 4, but compound 4 times (every 3 months)

$$\overbrace{\$10,000 + (.025)(\$10,000)}^{\text{after one period}}$$

$$\overbrace{+ (.025)(\$10,000 + (.025)(\$10,000))}^{\text{after two periods}}$$

after three periods

$$\left\{\begin{aligned} &+ (.025)(\$10,000 + (.025)(\$10,000) \\ &\quad + (.025)(\$10,000 + (.025)(\$10,000))) \end{aligned}\right.$$

after four periods

$$\left\{\begin{aligned} &+ (.025)(\$10,000 + (.025)(\$10,000) \\ &\quad + (.025)(\$10,000 + (.025)(\$10,000)) \\ &\quad + (.025)(\$10,000 + (.025)(\$10,000)) \\ &\quad + (.025)(\$10,000 + (.025)(\$10,000))) \end{aligned}\right.$$

This turns out to be

$$(\$10{,}000)(1+.025)^4$$

rate

$$= (\$10{,}000)\left(1+\frac{.01}{4}\right)^{4}$$

number of compounding periods

$$\approx \$11{,}038$$

After compounding n times in a year, we have

$$\boxed{\$10{,}000\left(1+\frac{.10}{n}\right)^{n}}$$

3) <u>Monthly</u>

$$\$10{,}000\left(1+\frac{.10}{12}\right)^{12}$$

$$\approx \$11{,}047.$$

It seems that compounding more times makes more money.

Check that $\left(1+\frac{.10}{n}\right)^n$ is increasing: sufficient to check for $\left(1+\frac{.10}{x}\right)^x$ where $x>0$.

Let $f(x)=\left(1+\frac{.10}{x}\right)^x$.

Take $f'(x)$, show it is always positive.

Use $e^{\ln}$ trick:

$$\left(1+\frac{.10}{x}\right)^x = e^{\ln\left(\left(1+\frac{.10}{x}\right)^x\right)}$$

$$= e^{x\ln\left(1+\frac{.10}{x}\right)}.$$

Now take the derivative:

$$f'(x) = e^{x\ln\left(1+\frac{.10}{x}\right)} \cdot \frac{d}{dx}\left(x\ln\left(1+\frac{.10}{x}\right)\right)$$

$$= \underbrace{\left(1+\frac{.10}{x}\right)^x}_{>0}\left(\ln\left(1+\frac{.10}{x}\right) + \underbrace{x\frac{d}{dx}\left(\ln\left(1+\frac{.10}{x}\right)\right)}_{\text{chain rule}}\right)$$

$$= \left(1+\frac{.10}{x}\right)^x\left(\ln\left(1+\frac{.10}{x}\right) + \overbrace{x\cdot\frac{1}{1+\frac{.10}{x}}\cdot\frac{.10}{-x^2}}\right)$$

$$= \left(1+\frac{.10}{x}\right)^x\left(\ln\left(1+\frac{.10}{x}\right) + \frac{(.10)\cancel{x}}{-x^{\cancel{2}}-(.10)\cancel{x}}\right)$$

$$= \left(1+\frac{.10}{x}\right)^x\underbrace{\left(\ln\left(1+\frac{.10}{x}\right) + \frac{.10}{-x-.10}\right)}_{>0}$$

That showed $\left(1+\frac{.10}{x}\right)^x$ is increasing.

Now let

x = number of compounding periods,

r = rate,

t = time (in years),

and $f(x) = \left(1+\frac{r}{x}\right)^{tx}$.

We showed f is increasing.

For a fixed r and a fixed t, what does it increase to?

$$\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^{tx} = \boxed{e^{rt}}$$

(continuous compounding)

Continuous compounding with principal A gives you

$\boxed{Ae^{rt}}$ dollars at rate r

over time t.

Discrete compounding gives

$$\boxed{A\left(1+\frac{r}{n}\right)^{nt}}$$

4) Mystery option:

continuous compounding.

After one year, you have

$$\$10{,}000\, e^{.10} \approx \$11{,}051,$$

which is definitely better than monthly.

Exponential Decay

Alexander Litvinenko was a Russian spy supposedly poisoned by thallium (half-life of 12.3 days).

Exponential decay:

$$\boxed{Se^{-rt}}$$ where

S = amount of material at time zero

r = rate of decay

t = time (days)

Given that Litvinenko had 10 mcg of thallium in his system initially, how much is there in his system after a week?

Find r: half-life is 12.3 days.

$$5\text{mcg} = e^{-r(12.3)} \cdot 10\text{mcg}$$

$$\frac{1}{2} = e^{-r(12.3)}$$ – take ln.

$$r = \frac{-\ln(1/2)}{12.3}$$

The answer:

$$10e^{7\left(\ln(1/2)/12.3\right)} \text{ mcg}$$

$$\approx \boxed{6.7403 \text{ mcg}}$$